PROPERTIES OF PLANE AND AXISYMMETRICAL STATIONARY FLOWS IN MAGNETOHYDRODYNAMICS

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This investigation deals with the plane and axisymmetrical flows of an infinitely conducting gas. The conditions div $\rho v = 0$ and div $H = 0$ in this case allow the introduction of the stream function $\varphi(x, y)$ and the magnetic field function $\chi(x, y)$. The equations of motion of the gas are transformed and stated in terms of the variables ψ and χ. The resulting equations under certain additional assumptions have integrals, of which one is analogous to the Bernoulli integral and the other does not have any correspondence in ordinary gas dynamics. In the case of orthogonality of the magnetic field and the velocity field the solution of the problem reduces to a linear partial differential equation of the second order. Some particular solutions are also considered.

1. Plane flows. Consider the steady plane flow of an ideal infinitely conducting fluid. We shall assume that the velocity v and the magnetic field intensity H have only two components v_x, v_y, H_x, H_y, and that all characteristics of the flow are independent of the z-coordinate.

Under these assumptions the equation of induction may be integrated

$$v_x H_y - v_y H_x = a = \text{const} \neq 0 \tag{1.1}$$

Because of the conditions div $\rho v = 0$ and div $H = 0$ we can define two functions $\psi(x, y)$ and $\chi(x, y)$ by the use of the following formulas

$$\rho v_x = -\frac{\partial \Psi}{\partial y}, \qquad \rho v_y = \frac{\partial \Psi}{\partial x}, \qquad H_x = -\frac{\partial \chi}{\partial y}, \qquad H_y = \frac{\partial \chi}{\partial x} \tag{1.2}$$

where the condition (1.1) transforms into the following:

$$\frac{\partial \Psi}{\partial x}\frac{\partial \chi}{\partial y} - \frac{\partial \Psi}{\partial y}\frac{\partial \chi}{\partial x} = D\begin{pmatrix}\Psi, \chi \\ x, y\end{pmatrix} = a\rho \tag{1.3}$$

The equations of conservation of momentum under the given assumptions have the form

$$\rho \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) - \rho v_y \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) + \frac{1}{4\pi} Hy \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = -\frac{\partial p}{\partial x} \qquad (1.4)$$

$$\rho \frac{\partial}{\partial y}\left(\frac{v^2}{2}\right) + \rho v_x \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{1}{4\pi} H_x \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = -\frac{\partial p}{\partial y} \qquad (1.5)$$

We shall transform this system in terms of the new independent variables ψ and χ. To do this let us project equations (1.4) onto the direction of the stream-line and the direction of the magnetic line. We obtain

$$v_x \frac{\partial p}{\partial x} + v_y \frac{\partial p}{\partial y} + \rho v_x \frac{\partial}{\partial x}\frac{v^2}{2} + \rho v_y \frac{\partial}{\partial y}\frac{v^2}{2} + \frac{1}{4\pi} a \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = 0 \qquad (1.6)$$

$$H_x \frac{\partial p}{\partial x} + H_y \frac{\partial p}{\partial y} + \rho H_x \frac{\partial}{\partial x}\frac{v^2}{2} + \rho H_y \frac{\partial}{\partial y}\frac{v^2}{2} + a \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = 0 \qquad (1.7)$$

Using the properties of Jacobians, we have

$$\frac{\partial \Psi}{\partial x} = \begin{vmatrix} \partial\psi/\partial x & \partial\psi/\partial y \\ 0 & 1 \end{vmatrix} = D\begin{pmatrix} \psi, & y \\ x, & y \end{pmatrix} = D\begin{pmatrix} \psi, & y \\ \psi, & \chi \end{pmatrix} D\begin{pmatrix} \Psi, & \chi \\ x, & y \end{pmatrix} = a\rho \frac{\partial y}{\partial \chi} \qquad (1.8)$$

$$\frac{\partial \Psi}{\partial y} = -\begin{vmatrix} \partial\psi/\partial x & \partial\psi/\partial y \\ 1 & 0 \end{vmatrix} = -D\begin{pmatrix} \psi, & x \\ x, & y \end{pmatrix} = -D\begin{pmatrix} \psi, & x \\ \psi, & \chi \end{pmatrix} D\begin{pmatrix} \psi, & \chi \\ x, & y \end{pmatrix} = a\rho \frac{\partial x}{\partial \chi} \qquad (1.9)$$

Analogously we find

$$\frac{\partial \chi}{\partial x} = -a \frac{\partial y}{\partial \chi}, \qquad \frac{\partial \chi}{\partial y} = -a\rho \frac{\partial x}{\partial \Psi} \qquad (1.10)$$

From (1.2) and (1.8) to (1.10) there follows

$$v_x = a \frac{\partial x}{\partial \chi}, \qquad v_y = -a\rho \frac{\partial y}{\partial \Psi}, \qquad H_x = -a\rho \frac{\partial x}{\partial \Psi}, \qquad H_y = -a\rho \frac{\partial y}{\partial \Psi} \qquad (1.11)$$

Expressions of the form

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} H_x \frac{\partial f}{\partial x} + H_y \frac{\partial f}{\partial y}$$

where $f(x, y)$ is an arbitrary function, may be transformed in the following way

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} = a \frac{\partial f}{\partial \chi}; \qquad H_x \frac{\partial f}{\partial x} + H_y \frac{\partial f}{\partial y} = -a\rho \frac{\partial f}{\partial \Psi} \qquad (1.12)$$

The expressions for the vorticity and the magnetic intensity stated in terms of the new coordinates ψ and χ by using equations (1.11), have the form

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \rho \frac{\partial v^2}{\partial \Psi} + \rho \frac{\partial}{\partial \chi}\left(\frac{\mathbf{v}, \mathbf{H}}{\rho}\right) \tag{1.13}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \rho\left[\frac{\partial}{\partial \Psi}(\mathbf{v},\mathbf{H}) + \frac{\partial}{\partial \chi}\frac{H^2}{\rho}\right] \tag{1.14}$$

Summarizing the results obtained, we have the system of equations

$$\frac{1}{\rho}\frac{\partial p}{\partial \chi} + \frac{\partial}{\partial \chi}\frac{v^2}{2} + \frac{\partial}{\partial \chi}\frac{H^2}{4\pi\rho} + \frac{1}{4\pi}\frac{\partial}{\partial \Psi}(\mathbf{v}, \mathbf{H}) = 0 \tag{1.15}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \Psi} - \frac{\partial}{\partial \Psi}\frac{v^2}{2} - \frac{\partial}{\partial \chi}\frac{(\mathbf{v}, \mathbf{H})}{\rho} = 0 \tag{1.16}$$

$$v_x H_y - v_y H_x = 0, \quad v_x = a\frac{\partial x}{\partial \chi}, \quad v_y = a\frac{\partial y}{\partial \chi}, \quad H_x = a\rho\frac{\partial y}{\partial \Psi}$$

$$H_y = -a\rho\frac{\partial y}{\partial \Psi} \tag{1.17}$$

It is easily seen, that when the scalar product of the velocity vector and the magnetic intensity does not depend on Ψ and the gas is isentropic, equation (1.15) is integrable

$$P + \frac{H^2}{4\pi\rho} + \frac{v^2}{2} = f_1(\psi) \quad \left(P = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{u^2}{\gamma - 1}, \ \gamma = \frac{c_p}{c_v}\right) \tag{1.18}$$

where u is the velocity of sound. The integral just derived is analogous to the Bernoulli integral in ordinary gas dynamics.

If the quantity $(\mathbf{v}, \mathbf{H})/\rho$ does not depend on χ and the gas is isentropic, equation (1.16) also may be integrated

$$P - \frac{v^2}{2} = f_2(\chi) \tag{1.19}$$

This integral does not have a counterpart in ordinary gas dynamics. It may be rewritten in the following way:

$$\frac{u^2}{\gamma - 1} = \frac{v^2}{2} + \frac{u_0^2}{\gamma - 1}$$

where u_0 is the velocity of sound at the point of the given magnetic line at which the velocity is equal to zero. In this form equation (1.19) shows that along a magnetic line the hydrodynamic pressure and the velocity of sound increase with increase of velocity. When the magnetic field and the velocity are orthogonal and the gas is isentropic, both integrals are valid simultaneously in the following way

$$\frac{u^2}{\gamma - 1} + \frac{H^2}{4\pi\rho} + \frac{v^2}{2} = f_1(\psi), \qquad \frac{u^2}{\gamma - 1} - \frac{v^2}{2} = f_2(\chi)$$

Here condition (1.1) becomes $vH = a$; using integrals (1.1), (1.18) and (1.19) quantities p, ρ, v, H may be expressed in terms of functions $f_1(\psi)$ and $f_2(\chi)$, which may be assumed to be known.

2. Plane steady gas flows in an orthogonal magnetic field. We shall investigate more in detail, flows in which the magnetic field is orthogonal to the velocity field. As was shown in the preceding section the integrals (1.18) and (1.19) remain valid. In addition to these relationships we have the condition of orthogonality and the condition (1.1)

$$v_x H_x + v_y H_y = 0, \qquad v_x H_y - v_y H_x = a \tag{2.1}$$

Hence

$$v_x = \frac{aH_y}{H^2}, \qquad v_y = -\frac{aH_x}{H^2} \tag{2.2}$$

Taking into account (1.2), we obtain

$$\frac{\partial\Psi}{\partial y} = -\frac{a\rho}{H^2}\frac{\partial\chi}{\partial x}, \qquad \frac{\partial\Psi}{\partial x} = \frac{a\rho}{H^2}\frac{\partial\chi}{\partial y} \tag{2.3}$$

or from (1.1)

$$\frac{\partial x}{\partial\chi} = -\frac{a\rho}{H^2}\frac{\partial y}{\partial\Psi}, \qquad \frac{\partial x}{\partial\Psi} = \frac{H^2}{a\rho}\frac{\partial y}{\partial\chi} \tag{2.4}$$

Hence, in this manner the problem is reduced to the solution of equation

$$\frac{\partial}{\partial x}\left(\frac{H^2}{a\rho}\frac{\partial\Psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{H^2}{a\rho}\frac{\partial\Psi}{\partial y}\right) = 0 \tag{2.5}$$

or

$$\frac{\partial}{\partial\chi}\left(\frac{H^2}{a\rho}\frac{\partial y}{\partial\chi}\right) + \frac{\partial}{\partial\Psi}\left(\frac{a\rho}{H^2}\frac{\partial y}{\partial\Psi}\right) = 0 \tag{2.6}$$

The quantity $\zeta = a\rho/H^2$ may be found from the integrals (1.18) to (1.19) and therefore it may be considered a known function of the quantities ψ, χ, $\zeta = \zeta(\psi, \chi)$.

In general equations (2.5) and (2.6) are complex but here we may confine our attention to the case where one of the functions $f_1(\psi)$ or $f_2(\chi)$ is constant. Then ζ becomes a function of one variable only. Consider the case when $f_1(\psi)$ = const. The quantities p, ρ, H and v depend only on the variable χ, consequently, they will be constant along any magnetic line. Here $\zeta = \zeta(\chi)$ and equations (2.3) are restated as follows:

$$\frac{\partial\Psi}{\partial y} = -\frac{\partial F}{\partial x}, \qquad \frac{\partial\Psi}{\partial x} = \frac{\partial F}{\partial y} \qquad \left(F(x, y) = \int_{\chi_0}^{\chi} \zeta d\chi\right) \tag{2.7}$$

Hence it is seen that the function $W(z) = \psi(x, y) + iF(x, y)$ is analytic. However, not every harmonic function is a solution to the given problem. Conditions (1.18) to (1.19) require that four out of five of the quantities p, H, v, ρ and χ must be functions of the fifth, therefore

$$\mathrm{grad}^2 \psi = \Phi(\chi) \quad \text{or} \quad D\begin{pmatrix} F, & \mathrm{grad}^2 \psi \\ x, & y \end{pmatrix} = 0 \tag{2.8}$$

Equation (2.8) may be transformed with the aid of (2.7) as follows

$$\frac{\partial \psi}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial y} \Big/ \frac{\partial \psi}{\partial x}\right] + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial \psi}{\partial x} \Big/ \frac{\partial \psi}{\partial y}\right] = 0$$

Assuming $(\partial\psi / \partial x) : (\partial\psi / \partial y) = -\rho v_y / \rho v_x = -\tan\theta = -z$, we obtain

$$\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = 0 \tag{2.9}$$

The general integral of equations (2.9) has the form

$$\Omega\,(xz - y,\, z) = 0 \tag{2.10}$$

where Ω is an arbitrary function of two variables. Hence only those harmonic stream functions $\psi(x, y)$ can be solutions of the given problem for which the tangent of the angle of inclination of the velocity satisfies equation (2.9) or (2.10). Consider the case where all the quantities p, ρ, H and v depend only on Ψ; this is the case when $f_2(\chi) = \text{const}$. By reasoning in the same way as above it may be shown that the process of solution is then reduced to finding the harmonic function $\chi(x, y)$, for which the tangent of the magnetic flux satisfies equation (2.9).

3. Radial flow of a conducting gas in an orthogonal magnetic field. In the simplest case the expression (2.10) may be represented in the form

$$xz - y = 0 \quad \text{or} \quad z = \tan\theta = y / x \tag{3.1}$$

Hence $\theta = \varphi$, where φ is a polar angle and, consequently, the stream lines are rays arising at the origin of the coordinate system. The lines of magnetic force are the circles $r = \text{const}$. The quantities v, H, ρ, p and χ are functions of r. Assuming

$$W = \Psi + iF = iALn\,(x + iy)$$

we obtain

$$\Psi = -A \tan^{-1}\frac{y}{x}, \qquad F = A \ln r \tag{3.2}$$

Hence

$$\rho v_x = \frac{Ax}{x^2+y^2}, \quad \rho v_y = \frac{Ay}{x^2+y^2}, \quad \rho v = \frac{A}{r}, \quad H = \frac{a\rho r}{A}\, H_x = \frac{a\rho y}{A}, \quad H_y = -\frac{a\rho x}{A} \tag{3.3}$$

When we substitute the stated expressions for these quantities into the condition (1.18), we obtain an equation which defines the density ρ as a function of radius r (equation (1.19) for the determination of function $f_2(\chi)$)

$$\frac{\gamma}{\gamma-1} c\rho^{\gamma-1} + \frac{A^2}{2}\frac{1}{\rho^2 r^2} + \frac{a^2 r^2 \rho}{4\pi A^2} = \text{const} \tag{3.4}$$

From (3.3) it follows that

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0 \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -\frac{a}{A}\left(2\rho + r\frac{d\rho}{dr}\right)$$

The given flow will be irrotational, but there will be a non-zero electric current in the gas. This solution was obtained by other methods in [1,2]. As may be shown easily, there exist two boundary circles $r = R_1$ and $r = R_2$ beyond which the solution may not be continued and on which the flow velocity quickly reaches the magneto-sound velocity. In the interval $R_1 < r < R_2$ there exist two types of flows, namely submagnetosonic and supermagnetosonic.

4. Plane vortical flow. Assume

$$\chi = -A \tan^{-1}\frac{y}{x}, \qquad f_2(\chi) = \text{const}$$

Then

$$H = \frac{Ax}{x^2+y^2}, \quad H_y = \frac{Ay}{x^2+y^2}, \quad H = \frac{A}{r}, \quad v = \frac{ar}{A}, \quad v_x = \frac{ay}{A}, \quad v_y = -\frac{ax}{A} \tag{4.1}$$

The gas rotates as a solid body in a radial magnetic field. Equation (1.19) may be used to determine the density

$$\frac{\gamma c}{\gamma-1}\rho^{\gamma-1} - \frac{a^2}{A^2} r^2 = \frac{u_0^2}{\gamma-1} \tag{4.2}$$

The pressure is determined from the condition that the flow be adiabatic. The solution is valid in the entire plane, except at the origin of the coordinate system.

5. Steady axisymmetrical flows of an infinitely conducting gas. Let us assume that the azimuthal components of the magnetic intensity and the velocity of flow are equal to zero and that none of the characteristics of flow depend on the angle φ in the cylindrical coordinate system. In this case we may introduce the stream function ψ and the magnetic force function χ:

$$\rho v_r = \frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad \rho v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad H_r = \frac{1}{r}\frac{\partial\chi}{\partial z}, \qquad H_z = -\frac{1}{r}\frac{\partial\chi}{\partial r} \tag{5.1}$$

The equation of induction is integrable here as it is in the plane case, and we obtain

$$r\,(H_r v_z - v_r H_z) = a = \mathrm{const} \tag{5.2}$$

As in the plane case, the equation of motion may be transformed into an equation in terms of variables ψ and χ. By reasoning in the same manner as in Section 1, we obtain

$$\frac{1}{\rho}\frac{\partial p}{\partial\chi} + \frac{\partial}{\partial\chi}\left(\frac{v^2}{2}\right) + \frac{1}{4\pi}\frac{\partial}{\partial\psi}(vH) + \frac{\partial}{\partial\chi}\left(\frac{H^2}{4\pi\rho}\right) = 0 \tag{5.3}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial\psi} - \frac{\partial}{\partial\psi}\left(\frac{v^2}{2}\right) - \frac{\partial}{\partial\chi}\left(\frac{vH}{\rho}\right) = 0, \qquad r\,(H_r v_z - H_z v_r) = a = \mathrm{const}$$

$$v_z = a\frac{\partial z}{\partial\chi}, \qquad v_r = a\frac{\partial r}{\partial\chi}, \qquad H_z = -a\rho\frac{\partial z}{\partial\psi}, \qquad H_r = -a\rho\frac{\partial r}{\partial\psi}$$

The integral (5.2) differs from the integral in the plane case in that its left side is multiplied by r. By making corresponding assumptions about the character of the functions $(\mathbf{vH})$ and $\mathbf{vH}/\rho$ and the barotropic dependence of the pressure on the density, the first two equations of (5.3) may be integrated and particular solutions are obtained which correspond to those obtained previously.

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